OPTIMAL DIVERSIFICATION IN ALLOCATION PROBLEMS

Ion Purcaru
The Bucharest Academy of Economic Studies, Romania
E-mail: ionpurcaru@yahoo.fr

Abstract

Because real choice is an important quality parameter for the retail sector, managers of retailing companies need an instrument to assist them in optimising the diversity of options provided to their customers. In this paper, using the weighted diversity index of Guiaşu and Guiaşu), some possibilities to obtain the optimal joint distributions of a probabilistic experiment and the corresponding optimal diversifications to a certain probabilistic system are proposed. These results can be used in ecology, in economy or in problems of allocation or of transportation as an example of diversity management.

Keywords: weighted diversity, principle of maximum diversity, diversification, optimal distributions.

JEL Classification: C02, C44, C61, C63

1. Quality management in retailing and consumer choice

Choice is an important constituent of the quality of retailing. The possibility to choose among a large variety of possibilities to satisfy a certain necessity has a distinct value according to the perception of the customer. The client of a retail shop will be less satisfied when buying a product that it is perfectly fitted to his needs if it is the only product in the offer, compared to the situation in which the same product is chosen from a multitude of variants.

Consumers' free choice is considered the engine of competition and an important lever of economic and social progress. By their choice as consumers, people fuel the market selection mechanism, and they implicitly determine winners and losers among producers. This the main perspective from which it is considered, by economic thinking, the consumer's choice.

In the field of microeconomic decisions the choices made by consumers are studied mainly in order to get information about preferences. Using this information in designing products and services increases their chances to be accepted by the market and represents a way to raise producers' financial performance. This practice insistently recommended by marketing specialists, may, indirectly, produce the effect of limiting, in fact, the possibility of choice, because it omits the gain in welfare due to choice itself.

When one consider the role of retailing, creating choice opportunities is usually mentioned as one of the main components of the value proposition of this segment of the supply chain (Eurostat, 2001). Apparently, consumers will prefer shopping there where they have a larger choice. Therefore, the leaders of retail businesses have an interest to play on this aspect in order to increase sales and grow the business. In the literature, it is mentioned a difference between theoretical diversity and “real” choice, the former being characterized by the fact the product variety do not follow the real motivational factors of consumer choices (Clarke, I. et al., 2004)

Consequently there is an interest of retailers to manage the issue of product diversity. Despite this fact, there are very few instruments for modeling and optimizing this phenomenon. This is why I considered useful to propose a mathematic instrument that will possibly improve significantly the capacity of managers of retail companies to improve the quality of their service in a way which is easily assessable and favorably impacting on the customers’ preference.

Using the weighted diversity index of Guiaşu and Guiaşu ([5]), some possibilities to obtain the optimal joint distributions of a probabilistic experiment and the corresponding optimal diversifications to a certain probabilistic system are proposed during the following chapters.

2. Introduction to the model

Let us consider an arbitrary discrete joint probabilistic experiment $(X,Y)=[(x_i,y_j);(\pi_{ij})]$, where the pairs (x_i,y_j) are the outcomes of the component probabilistic experiments X and Y, respectively, and (π_{ij}) is the joint occurrence probability of outcomes x_i in X and outcomes y_j in Y, for which we have:

$$\pi_{ij} \geq 0, (1 \leq i \leq m; 1 \leq j \leq n), \ \sum_{i=1}^{m}\sum_{j=1}^{n}\pi_{ij} = 1 \tag{1}$$

In this situation, the numbers $\pi=(\pi_{ij})$ represent the *joint probability distribution* of the couple (X,Y) and the numbers $p=(p_i)$ and $q=(q_j)$, given by the following relations:

$$p_i = \sum_{j=1}^{n}\pi_{ij}, \quad 1 \leq i \leq m; \qquad q_j = \sum_{i=1}^{m}\pi_{ij}, \ 1 \leq j \leq n; \tag{2}$$

represent the *marginal probability distributions* of the couple (X,Y). Sometimes, the joint probability distribution of the couple (X,Y) is given but other times this distribution must be determined. In this paper we get some solutions of this problem. Let us consider the strict positive numbers $u=(u_{ij})$ representing the weights assigned to the pairs $[(x_i,y_j),(\pi_{i,j})]$ of the couple (X,Y). According to Guiaşu and Guiaşu ([5]), the *weighted diversity index* corresponding to the joint discrete probability distribution $\pi=(\pi_{ij})$, with the weigts $u=(u_{ij})$, is given by the number:

$$D(\pi,u) = \sum_{i=1}^{m}\sum_{j=1}^{n}u_{ij}\pi_{ij}(1-\pi_{ij}) \tag{3}$$

and for ***n=1***, the *weighted diversity index* corresponding to the discrete probability distribution $p=(p_i)$, with the weights $w=(w_i)$, is given by the number:

$$D(p,w)=\sum_{i=1}^{m} w_i p_i(1-p_i) \tag{4}$$

If all $w_i=1$, then (4) is just the diversity index of Gini, Onicescu and Simpson of the simple discrete distribution $p=(p_i)$, better known as the Simpson diversity index (see Guiasu [5]), namelly:

$$D(p)=1-\sum_{i=1}^{m} p_i^2 \tag{5}$$

Let us consider the real numbers $z=(z_{ij})$ in a certain connection with the results of the joint experiment (X,Y) and an associated mean value of the experiment denoted by $F(\pi,z)$ and given by the relation:

$$F(\pi,z)=\sum_{i=1}^{m}\sum_{j=1}^{n}\pi_{ij}z_{ij} \tag{6}$$

which, sometimes, can be fixed (or pre-established) as a practical constraint by the type:

$$F(\pi,z)=Z_0\,(known),\quad \min\{z_{ij}\}\le Z_0\le \max\{z_{ij}\} \tag{7}$$

Remarks. If we consider the following unknown real numbers $x_{ij}\ge 0$, with the properties

$$\sum_{j=1}^{n} x_{ij}=a_i\,(known),\ \sum_{i=1}^{m} x_{ij}=b_j\ (known),\ \sum_{i=1}^{m} a_i=\sum_{j=1}^{n} b_j=T$$

with a certain economical signification and if we denote by:

$$\pi_{ij}=\frac{x_{ij}}{T}\,(unknown),\ p_i=\frac{a_i}{T}\,(known),\ q_j=\frac{b_j}{T}\,(known)$$

for all pairs (i, j), then we have a problem of the determination of the numbers (x_{ij}) or of their equivalents (π_{ij}), such as in the situations of the classical or the special problems of allocation, where *T* signify the total quantity (see [3]). If the distribution $\pi=(\pi_{ij})$ is determined, then the allocation solution is given by the numbers $(x_{ij})=(\pi_{ij}\times T)$. For a classical allocation problem, the number x_{ij} represent the allocated quantity between the partners i and j. Also, in the case of a problem of allocation, if the numbers $z=(z_{ij})$ are the costs (profits) per unity, then $F(\pi,z)$ signify the total mean cost (profit) per unity implying a total cost (profit) $C=F(\pi,u)\times T$ and as a result, we search the numbers (π_{ij}) minimizing (or maximizing) $F(\pi,z)$.

As an extension of the Principle of Maximum Entropy (PME), which assert that: “from the set of all probability distributions compatible with one or several mean values of one or several random variables, choose the one that maximize the Shannon entropy”, Guiaşu ([5]) introduced the Principle of Maximum Diversity (PMD), according to which: “from the set of all probability distributions compatible with one or several mean values of one or several random variables, choose the one that maximize the diversity index”. Thus, applying the PMD, we can obtain, in certain conditions, the joint probability distributions which maximize the corresponding weighted diversity degree. Also, we can obtain the joint probability distributions which satisfy some constraints, preserve a certain degree of diversity of this distribution and maximize or minimize an assigned numerical expression of this experiment. These cases represent, for example, two situations of diversification in two special problems of (allocation) and, in the following, some corresponding results of these interesting situations are presented. The searched optimal distribution represents a certain optimal diversification category for some economical or ecological systems.

Notice. In the following presentation the notations $\sum_i \sum_j (*) = \sum_{i=1}^{m} \sum_{j=1}^{n} (*)$ are ewerywhere assumed.

3. Optimal diversification with maximum diversity degree.

We search the joint probability distributions which are compatible with certain constraints, one constraint is (7) signifying a fixed profit, and which maximize the weighted diversity degree (3).

Proposition 1. *The joint probability distribution* $\pi^0 = (\pi_{ij}^0)$ *compatible with the constraints (7) and (8)*

$$\sum_i \pi_{ij} = q_j \text{ (known)}, \quad \sum_j \pi_{ij} = p_i \text{ (known)}, \ 1 \le i \le m, \ 1 \le j \le n \tag{8}$$

which maximize the weighted diversity (3) is given by the numbers:

$$\pi_{ij}^0 = \frac{1}{2}\left[1 + (a_i + b_j + cz_{ij})v_{ij}\right] = \pi_{ij}^0(Z_0)\,;\, v_{ij} = \frac{1}{u_{ij}}\,; 1 \le i \le m, \ 1 \le j \le n \tag{9}$$

where, the parameters (a_i, b_j, c) *represent the solution of the following system*

$$\sum_{i=1} (a_i + b_j + cz_{ij})v_{ij} = 2q_j - m;\ 1 \le j \le n \tag{10.1}$$

$$\sum_{j=1} (a_i + b_j + cz_{ij})v_{ij} = 2p_i - n;\ 1 \le i \le m \tag{10.2}$$

$$\sum_{i=1} \sum_{j=1} (a_i + b_j + cz_{ij})v_{ij}z_{ij} = 2Z_0 - Z \tag{10.3}$$

for those numbers (u_{ij}), (p_i), (q_j), (z_{ij}) *and* Z_0 *for which* $(\pi_{ij}^0) \geq 0$, *with the maximum diversity:*

$$D_{\max} = D(\pi^0, u) = \sum_i \sum_j u_{ij} \pi_{ij}^0 (1 - \pi_{ij}^0) = \frac{1}{4}\left[U - \sum_i \sum_j (a_i + b_j + cz_{ij})^2 v_{ij}\right] \quad (11)$$

where $U = \sum_i \sum_j u_{ij}$ *and* $Z = \sum_i \sum_j z_{ij}$.

Proof. We want to solve a nonlinear program defined by the function (3) and the constraints (7) and (8). Using the method of Lagrange multipliers [7] we get the formulas (9)-(11) and the Proposition is proved.

Remarks. The expression (9) is a parametrical optimal probability distribution of the allocation model (the parameter Z_0 is the fixed or pre-established profit or cost).

4. Optimal diversification with given diversity degree.

Let us consider the number D_0 as a pre-established diversity degree for the weighted diversity (3), namely,

$$D(\pi, u) = D_0(known); \;\; 0 \leq D_0 \leq D_{\max} = \frac{1}{4}\left[U - \frac{(mn-2)^2}{V}\right] \leq \frac{U}{4} \quad (12)$$

where, the maximum value $D_{\max}$ is realized for the joint probability distribution

$$\pi_{ij} = \frac{1}{2}\left[1 + \frac{2 - mn}{V} v_{ij}\right], v_{ij} = \frac{1}{u_{ij}}; 1 \leq i \leq m, \; 1 \leq j \leq n \quad (13)$$

only for those numbers (u_{ij}) for which $V \geq (mn-2)\max\{v_{ij}\}$, where $V = \sum_i \sum_j v_{ij}$.

We search the joint probability distributions which are compatible with certain constraints, one constraint is (13) signifying a pre-established diversity degree, and which optimize the mean value (6).

Proposition 2. *The joint probability distribution* $\pi^0 = (\pi_{ij}^0)$ *compatible with the constraints (8) and (12) which maximize or minimize the mean value (6) is given by:*

$$\pi_{ij}^0 = \frac{1}{2}\left[1 + (a_i + b_j + cz_{ij})v_{ij}\right]; c \neq 0, v_{ij} = \frac{1}{u_{ij}}; 1 \leq i \leq m, \; 1 \leq j \leq n \quad (14)$$

where, the parameters (a_i, b_j, c) *represent the solution of the following system*

$$\sum_{i=1} (a_i + b_j + cz_{ij})v_{ij} = 2q_j - m; \;\; 1 \leq j \leq n \quad (15.1)$$

$$\sum_{j=1} (a_i + b_j + cz_{ij})v_{ij} = 2p_i - n; \;\; 1 \leq i \leq m \quad (15.2)$$

$$\sum_{i=1}\sum_{j=1}(a_i + b_j + cz_{ij})^2 v_{ij} = U - 4D_0 \ ; \ U - 4D_0 > 0; \qquad (15.3)$$

only for those numbers (u_{ij}), (z_{ij}) *and* D_0 *for which* $(\pi^0_{ij}) \geq 0$ *and as a result,*

1) if c *>0, then the distribution (14) maximize* $F(\pi,u)$ *;*

2) if c *<0, then the distribution (14) minimize* $F(\pi,u)$

and the corresponding optimal value of $F(\pi,u)$ *is given by:*

$$F_{opt} = F(\pi^0, u) = \sum_i\sum_j \pi^0_{ij} z_{ij} = \frac{1}{2}\left[Z + \sum_i\sum_j (a_i + b_j) v_{ij} z_{ij} + cB \right] \qquad (16)$$

with $F_{opt} = F_{\max}$ *(if* c *>0) and* $F_{opt} = F_{\min}$ *(if* c *<0), while* $B = \sum_i\sum_j v_{ij} z^2_{ij}$.

Proof. We want to solve a nonlinear program defined by the function (6) and the constraints (8) and (12). Using the method of Lagrange multipliers we obtain the formulas (14)-(16) and the Proposition is proved.

Remarks. The expressions (14) is a parametrical optimal probability distributions of the allocation model (the parameter D_0 is the fixed diversity degree) of the allocation model. If $n = 1$ then all the corresponding formulas of the Propositions 1-2 are valid for the experiment $X = [x_i; p_i]$ too.

5. Numerical examples and some conclusions.

Example. Let us consider an allocation situation with 2 offers $(m = 2)$ and 3 demands ($n = 3$), with the individual offers and respectively demands (the marginal probability distributions) $(p_i) = (2/5;3/5)$ and $(q_j) = (2/10;3/10;5/10)$, with the utilities $(u_{ij}) = \begin{pmatrix} 1/2 & 1/3 & 1/4 \\ 1/4 & 1/2 & 1/3 \end{pmatrix}$, with the numbers (costs or profits per unity) $(z_{ij}) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and with the total capital of allocation $T = 749280$ units.

We have $1 \leq Z_0 \leq 3$ and $0 < D \leq 0{,}41(6)$.

We search the optimal allocation for the capital T and the corresponding diversifications of this allocation in the conditions of the allocation models introduced by the Propositions 1 and 2.

Solutions. Remarks. In the conditions of a classical allocation, the problem has a single optimal solution X_1^0 in the case of minimization and, respectively, X_2^0 in the case of maximization, namely:

$$X_1^0 = \begin{pmatrix} 149856 & 149856 & 0 \\ 0 & 74928 & 374640 \end{pmatrix} \text{ and } X_2^0 = \begin{pmatrix} 0 & 0 & 299712 \\ 149856 & 224748 & 74928 \end{pmatrix}$$

with unitary mean cost $Z_0 = 1{,}3$ u.m., total cost $C_0 = 974064 u.m.$ and diversity $D = 0{,}535$ and with mean unitary profit $Z_0 = 2{,}5$u.m., total profit $C_0 = 1873200 u.m.$ and diversity $D = 0{,}328(3)$.

a) In the conditions of Proposition 1, writing and solving the corresponding system (10.1)-(10.3) then, without the laborious detailed calculus, finally, we get the following parametrical distribution

$$(\pi_{ij}^0(Z_0)) = \frac{1}{3360}\begin{pmatrix} 1977 - 810Z_0 & 246 - 60Z_0 & 870Z_0 - 879 \\ 810Z_0 - 1305 & 60Z_0 + 762 & 2559 - 870Z_0 \end{pmatrix}$$

which is admissible for $\frac{435}{270} \le Z_0 \le \frac{659}{270}$. Particularly, for $Z_0 = 2$ (a fixed total mean profit per unity of allocation implying a fixed total mean profit $C_0 = T \times Z_0 = 1498560$ units of allocation operation), we get the optimal distribution

$$(\pi_{ij}^0(Z_0 = 2)) = \frac{1}{1120}\begin{pmatrix} 119 & 42 & 287 \\ 105 & 294 & 273 \end{pmatrix}$$

and the corresponding optimal solution of allocation

$$[x_{ij}(Z_0 = 2) = \pi_{ij}^0(Z_0 = 2) \times T] = \begin{pmatrix} 70611 & 28098 & 192003 \\ 70245 & 196686 & 182637 \end{pmatrix}$$

with maximum diversity degree of this operation $D_{\max} \cong 0{,}292285$.

Remarks. We observe that $Z_0 = 1{,}3$ and $Z_0 = 2{,}5 \notin \left(\frac{435}{270}; \frac{659}{270}\right)$ and the solutions of the classical allocation does not admissible in the conditions of Proposition 1.

b) In the conditions of Proposition 2, writing and solving the corresponding system (15.1)-(15.3) then, without the laborious detailed calculus, finally, we get the following parametrical distribution

$$[\pi_{ij}^0(c(D_0))] = \frac{1}{2230}\begin{pmatrix} 431 - 3240c & 98 - 240c & 363 + 3480c \\ 15 + 3240c & 571 + 240c & 752 - 3480c \end{pmatrix}$$

which is admissible for $-\frac{15}{3240} \le c \le \frac{431}{3240}$, $c \ne 0$, where the parameter $c = c(D_0)$ is the solution of the processed equation (15.3) which has the final aspect

89913600 c^2 +29837400 D_0 =8875177, with the significations of the Proposition 2 for the values $c<0$ and $c>0$.

2.1. Particularly, for $c=1/8=0{,}125$ (implying a fixed degree of diversity $D_0 \cong 0{,}250366$), we get the optimal distribution of allocation which maximize the total mean profit per unity $F(\pi,u)$

$$[\pi_{ij}^0(c(D_0)=0{,}125)]=\frac{1}{2230}\begin{pmatrix}26 & 68 & 798\\ 420 & 601 & 317\end{pmatrix} \text{ with } F_{\max}=\frac{1067}{446}$$

and the corresponding optimal solution of allocation.

$$\left(x_{ij}[c(D_0)=0{,}125]=\pi_{ij}^0[c(D_0)=0{,}125]\times T\right)=\begin{pmatrix}8736 & 22848 & 268128\\ 141120 & 201936 & 106512\end{pmatrix}$$

with the maximum total mean profit of this operation $P_{\max}=F_{\max}\times T=1792560$.

2.2. Particularly, for $c=-1/240$ (implying a fixed degree of diversity $D_0 \cong 0{,}297399$), we get the optimal distribution of allocation which minimize the total mean cost per unity $F(\pi,u)$

$$[\pi_{ij}^0(c(D_0)=-\frac{1}{240})]=\frac{1}{4460}\begin{pmatrix}889 & 198 & 697\\ 3 & 1140 & 1533\end{pmatrix} \text{ with } F_{\min}=\frac{3409}{2230}$$

and the corresponding optimal solution of allocation

$$\left(x_{ij}[c(D_0)=-\frac{1}{240}]=\pi_{ij}^0[c(D_0)=-\frac{1}{240}]\times T\right)=\begin{pmatrix}149352 & 33264 & 117096\\ 504 & 191520 & 257544\end{pmatrix}$$

with the minimal total mean cost of this operation $C_{\min}=F_{\min}\times T=1145424$.

6. Final remarks

Generally, the mathematical theoretical presentation is very uncomfortable but it is very necessary too! The theoretical results of the Propositions 1-2 and the presented associated numerical examples show the possibilities to use the diversity measure, in certain conditions, generally, for the determination of the corresponding joint probability distributions and, particularly, for the solving of some special problems of allocation or of transportation. Sometimes, the results of these propositions can be considered as decisional models in the management of diversity with an adequate interpretation of the problem and of the solutions.

The model that is presented here can be used successfully with different other problems of diversity as the one suggested by Spanish scholars relating to the management of variety as a mean to improve business performance in areas where the business density is neither high nor low [3].

The author is committed to work with retailers in order to refine the model and prove its applicability in real circumstances. This approach represents, in fact the next stage of the research.

References

1. Blackmore, J.; Marenghi, C. (editors) (2008): *The PERFORMANCE Manager. Proven Strategies for Turning Information into Higher Business Performance for Retail*, Cognos Press, 2008
2. Clarke, I. et al. (2004): *Retail competition and consumer choice: contextualising the ‘‘food deserts’’ debate*, International Journal of Retail & Distribution Management, Volume 32 . Number 2 . 2004 . pp. 89-99
3. Esteban-Bravo, M.; Múgica, J.M.; Vidal-Sanz, J. (2006): *Do business density and variety determine retail performance?* Universidad Carlos III de Madrid, Working Paper 06-58, Business Economics Series 17, October 2006
4. Eurostat (2001): *Distributive trades in Europe – 2001*, Office for Official Publications of the European Communities, Luxembourg, 2001
5. Guiaşu, R. C. and Guiaşu, S. *Entropy in Ecology and Ethology*. Nova Scie. Publish., New York, 2003.
6. Guiaşu, S**.** *Information theory with applications*. McGraw-Hill, New York, 1977.
7. Purcaru, I. *Matematici generale şi elemente de optimizare.* Editura Economică, Bucureşti, 2004.